4.2

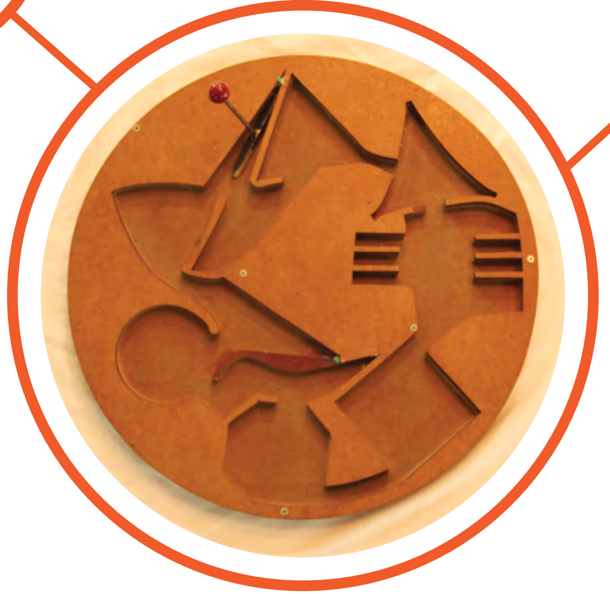
Kakeya Needle Problem

Imagine having to move a ladder, or a large stick, from one room to another or suppose one is trying to park a car in a tight spot. Most often in such a situation one has to navigate through a rather cramped space. Both tasks require a great deal of ingenuity and dexterity.

In the year 1917, a Japanese mathematician S. Kakeya considered the problem of finding the area of the smallest convex set in the plane inside which a needle of unit length can be reversed, i.e., turned around through 180º. (A "convex set" is nothing but an area which has no "spikes" e.g. a circle or a triangle is convex whereas a star in *not* convex.)

For example, one easy possibility for a convex set inside which a needle of unit length can be rotated through 180º is a semi-circle of radius 1 whose area is $\pi/2$. To do better than this, consider a circle of diameter 1; placing the needle along a diameter and rotating it through 360º around the centre gives us a set whose area is $\pi/4 = 0.78$. To do even better than this, consider an equilateral triangle ABC of height 1 and suppose that the needle lies on the side AB with its tip at vertex B; then, rotating the needle around the tip by 60º, sliding it backwards on the side BC, rotating around C by 60º, sliding it forward along side AC and finally rotating it through 60º around the vertex A and sliding it backwards along AB one can see that the needle has been reversed inside the triangle ABC. We have done better because the area of this triangle is $1/\sqrt{3} = 0.58$, which is smaller than the area of the circle of diameter equal to 1.

Indeed, Kakeya conjectured that an equilateral triangle of unit height is the smallest such set; he also observed that if the con-



vexity condition is dropped, that is, if we allow for shapes such as stars or octopus, with “spikes,” then a smaller set is possible. The more interesting question of finding the smallest set (without the convexity condition) in which to turn the needle came to be known as the *Kakeya needle problem*.

Let us first exhibit a non-convex set, of area smaller than that of an equilateral triangle of unit height, inside which one can reverse a unit needle. This can be achieved inside a three-cornered hypocycloid ABC, one of the shapes in the exhibit tray. It is possible to reverse the needle inside this region in a manner similar to the procedure followed in the case of the equilateral triangle of unit height. Here, start by placing a needle inside the hypocycloid with its tip at B. Then slide it tangentially along the curve BC until the other end of the needle is now tangential at C. Next, follow this procedure along the curve AC until the needle sits with its tip tangential at A. Finally, repeat the same procedure once again so that the other end of the needle sits tangentially at B thereby moving back to its original position with its ends switching positions as desired. The area of this hypocycloid ABC swept by the needle turns out to be equal to $\pi/8 = 0.39$, which is smaller than the area of the triangle ABC discussed above.

The conjecture with the convexity assumption on the set was proved in 1921 by J. Pál. The resolution of the Kakeya needle problem in 1928, without the convexity assumption, by Besicovitch has a curious history. Working on an entirely different problem, Besicovitch had constructed, in 1919, a certain set that

contains a unit segment in every direction. It was realized in 1928 that a simple modification to the Besicovitch set yielded a solution to the Kakeya problem that there exists sets of arbitrarily small area inside which it is possible to reverse a unit segment. How this solution was arrived at was indicated in the associated short computer animation. (This is available on the website.)

Where? Some of the techniques used in Besicovitch’s solution are exactly similar to the methods used in reversing of a vehicle in a cramped space. But this exhibit also illustrates that some very simple looking questions may lead to very rich mathematics that eventually is then used in many other areas of mathematics itself and in applications.